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Low-temperature equilibrium states of ferromagnetic lattice systems

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Abstract. A complete description of all periodic low-temperature Gibbs states of ferromagnetic classical lattice gas models with a finite but arbitrary number of different particles is obtained.

1. Introduction

We investigate the low-temperature behaviour of ferromagnetic classical lattice gas models with a finite but arbitrary number of different particles. At each site of the lattice there is a variable which can take on a finite number of values. One may think of these as different species which can occupy the lattice sites or $2l+1$ orientations of a spin-l particle. The particles or spins interact through many-body potentials. A configuration of particles with a minimal potential energy per lattice site is called a ground-state configuration. In the case of models which have a finite number of periodic ground-state configurations and satisfy the so-called Peierls condition (the creation of an 'island' of one periodic ground-state configuration in a 'sea' of another ground-state configuration leading to an increase of energy which is greater than some fixed positive constant times the length of the boundary of the 'island') there is a complete theory due to Pirogov and Sinai [1, 2] (see also the review article by Slawny [3]) where the phase diagram at low temperatures is obtained by perturbation of the zero-temperature phase diagram. In particular, the number of extremal periodic Gibbs states is equal to the number of periodic ground-state configurations.

The goal of this paper is to describe all low-temperature periodic Gibbs states for a large class of systems (ferromagnetic models) with an infinite number of periodic ground-state configurations. We generalise the results of Holsztyński and Slawny [3-5]. They described completely all periodic Gibbs states of translation-invariant ferromagnetic spin-$\frac{1}{2}$ models. In such systems, at each site of the lattice there is a spin variable which can take on just two values: 1 and -1 (spin up or spin down). Many-body interactions generalise that of the ferromagnetic Ising model: all coupling constants are negative.

Here we consider the following generalisation. Each site of the lattice can be occupied by one of the finite number of different species. To use the technique developed by Holsztyński and Slawny we make the set of all different particles into a

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finite Abelian group. Although one may think of a finite cyclic group, by discussing an arbitrary finite Abelian group case we enlarge the family of ferromagnetic models. The finite products of characters of this group at different lattice sites constitute the bonds of interaction between particles which occupy these lattice sites. We restrict ourselves entirely to the ferromagnetic case, i.e. negative coupling constants. One of the models which fits into our scheme is the ferromagnetic Potts model with m components, where the group is Z_m, the cyclic group of order m. The phase structures of general Z_m models was studied recently by Fröhlich and Spencer [6]. Z_m models were also used by Gruber *et al* [7] to investigate higher-spin systems. In the case of a cyclic group on a simple cubic lattice the complete results as in [4] were obtained in [8]. The case of an arbitrary lattice is more complicated. The energy of the ground-state configuration can be invariant under a local change. Spin-$\frac{1}{2}$ models on arbitrary lattices were investigated by Slawny [3, 5]. Here we generalise his results to the case of a finite number of different particles.

It is known [9-11] that to determine the periodic low-temperature Gibbs states of ferromagnetic systems it is enough to know the unbroken part, $\mathscr{S}^+$, of the group of all transformations, $\mathscr{S}$, which leave the interaction invariant. In particular, the number of extremal periodic Gibbs states—pure phases—is equal to $|\mathscr{S}/\mathscr{S}^+|$ at low temperatures. $\mathscr{S}$ can be identified with the ground-state configurations of the model so we can have a finite number of pure phases even in the case of an infinite number of periodic ground-state configurations. In fact, in two-dimensional models there is always a finite number of pure phases [3].

Here we present the explicit expression for $\mathscr{B}^+$, the group orthogonal to $\mathscr{S}^+$, in terms of the bonds of the system. In § 2 we introduce the notation and describe the model. For a more detailed description the reader should refer to [8]. In § 3 we generalise the main theorem from [5] to the finite Abelian group case. It relates the system to its reduced version. The reduced system has the so-called decomposition property so $\mathscr{B}^+$ of this system is explicitly expressed in terms of its bonds: $\mathscr{B}^+ = \mathrm{cl}\, A$ [5, 8]. Using the theorem one can find $\mathscr{B}^+$ for the original system. Section 4 contains the generalised construction of the reduced system. In § 5 an effective method of reduction in certain situations is presented. Section 6 contains some examples. We count the pure phases in certain models which have an infinite number of periodic ground-state configurations.

2. Notation and description of the model

2.1. *Configuration space*

By the lattice $\mathbb{L}$ is meant any $\mathbb{Z}^\nu$-invariant discrete subset of $\mathbb{R}^\nu$. A finite Abelian group $\mathscr{G}$ is placed at each site of the lattice.

$$\mathscr{X} = \times_{i \in \mathbb{L}} \mathscr{G} \tag{2.1}$$

is the configuration space of the system.

For $A \in \mathscr{X}$, $\mathrm{pr}_{\{a\}} A = A(a)$, $a \in \mathbb{L}$. If $\mathscr{G}$ is equipped with the discrete topology, then $\mathscr{X}$ becomes a compact Abelian group with the product topology.

$$\mathscr{X}_\Lambda = \bigoplus_{i \in \Lambda} \mathscr{G} \tag{2.2}$$

is a finite volume configuration space, where Λ is any finite subset of $\mathbb{L}$ and

$$\mathscr{X}_f = \bigoplus_{i \in \mathbb{L}} \mathscr{G}. \tag{2.3}$$

We provide $\mathscr{X}_f$ with the discrete topology. The group dual to $\mathscr{X}$ is isomorphic to $\mathscr{X}_f$:

$$\hat{\mathscr{X}} \simeq \mathscr{X}_f. \tag{2.4}$$

If $A \in \mathscr{X}_f$, then we write $\hat{A}$ for the corresponding element in $\hat{\mathscr{X}}$.

2.2. *Interaction*

The Hamiltonian in a finite volume, H_Λ, is a real, negative definite and translation invariant function on $\mathscr{X}_\Lambda$. This means that the Fourier decomposition of H_Λ is

$$H_\Lambda = -\sum_{B \in \mathscr{X}_\Lambda} J(B)\hat{B} \tag{2.5}$$

where $J(B) \geq 0$, $J(B) = J(B^{-1})$ and if B_2 can be obtained from B_1 by a translation, then $J(B_1) = J(B_2)$. The family of bonds is defined as

$$\mathscr{B} = \{B \in \mathscr{X}_f : J(B) > 0\}. \tag{2.6}$$

We assume that there is a finite fundamental family of bonds, $\mathscr{B}_0$, such that any element of $\mathscr{B}$ can be obtained in a unique way by a translation of an element from $\mathscr{B}_0$. Let $K(B) = \beta J(B)$, where β is the inverse temperature. Sometimes we refer to K as the interaction of a system.

2.3. *Gibbs states*

Let $e_{\mathscr{G}}$, the identity of the group $\mathscr{G}$, be placed everywhere outside Λ. With this as a boundary condition, a finite-volume Gibbs state can be constructed. It is denoted traditionally by ρ_Λ^+.

The Gibbs state ρ^+ can be constructed as a limit of ρ_Λ^+ when $\Lambda \to \mathbb{L}$. ρ^+ is a translation invariant state, extremal in the set of all Gibbs states and therefore mixing.

The following definitions are standard:

$$\mathscr{A}\text{—a subgroup of } \mathscr{X}_f \text{ generated by } \mathscr{B} \tag{2.7a}$$

$$\mathscr{B}^+ = \{A \in \mathscr{X}_f : \rho^+(\hat{A}) > 0\} \tag{2.7b}$$

$$\mathscr{S} = \{A \in \mathscr{X} : \hat{B}(A) = 1 \text{ for any } B \in \mathscr{B}\} \tag{2.7c}$$

$$\mathscr{S}^+ = \{G \in \mathscr{S} : \rho_G^+(\hat{A}) \equiv \hat{A}(G) \cdot \rho^+(\hat{A}) = \rho^+(\hat{A}) \text{ for any } A \in \mathscr{X}_f\}. \tag{2.7d}$$

The number of extremal periodic Gibbs states—pure phases—is equal to $|\mathscr{S}/\mathscr{S}^+|$ at low temperatures. $\mathscr{S}/\mathscr{S}^+$ is isomorphic to the group dual to $\mathscr{B}^+/\mathscr{A}$, hence $|\mathscr{S}/\mathscr{S}^+| = |\mathscr{B}^+/\mathscr{A}|$.

2.4. *Contours*

Denote:

$$\mathscr{M} = \times_{B \in \mathscr{B}} \mathbb{Z}_{|B|} \tag{2.8}$$

$$\mathscr{M}_f = \bigoplus_{B \in \mathscr{B}} \mathbb{Z}_{|B|}. \tag{2.9}$$

If each cyclic group is equipped with the discrete topology, then $\mathcal{M}$ becomes a compact Abelian group with the product topology. $\mathcal{M}_f$ with the discrete topology is a locally compact Abelian group. Both $\mathcal{M}$ and $\mathcal{M}_f$ and also $\mathcal{X}$ and $\mathcal{X}_f$ are $\mathbb{Z}_m[\mathbb{Z}^\nu]$-modules, where $\mathbb{Z}_m[\mathbb{Z}^\nu]$ is the group ring of all functions from $\mathbb{Z}^\nu$ to $\mathbb{Z}_m$ with a finite support. Two useful module homomorphisms can be constructed. Let

$$\gamma(X)=(\hat{B}(X))_{B\in\mathcal{B}} \qquad X\in\mathcal{X}. \tag{2.10}$$

Then it can be written: $\gamma(X)=\alpha\in M$ where $\hat{B}(X)=\exp[2\pi i\alpha(B)/|B|]$, so $\gamma:\mathcal{X}\to\mathcal{M}$.

Let now $\alpha\in\mathcal{M}$,

$$\varepsilon(\alpha)=\sum_{B\in\mathcal{B}}\alpha(B)B \tag{2.11}$$

so $\varepsilon:\mathcal{M}\to\mathcal{X}$. The sum converges in the topology of $\mathcal{X}$ because the interaction is of a finite range. It is easy to see that both γ and ε are continuous module homomorphisms. Let $\Gamma_f=\gamma(\mathcal{X}_f)$. Elements of Γ_f are called contours.

2.5. *Bicharacters on $\mathcal{X}\times\mathcal{X}_f$ and $\mathcal{M}\times\mathcal{M}_f$*

It is known that $\mathcal{X}$, $\mathcal{X}_f$ and $\mathcal{M}$, $\mathcal{M}_f$ are mutually dual groups. For $X\in\mathcal{X}$ and $Y\in\mathcal{X}_f$ we have

$$\langle X, Y\rangle\equiv\hat{Y}(X)=\prod_{a\in L}\exp\left[2\pi i\left(\sum_{i=1}^{r}X_i(a)Y_i(a)\right)(|\mathcal{G}_i|)^{-1}\right] \tag{2.12}$$

where $\mathcal{G}=\bigoplus_{i=1}^{r}\mathcal{G}_i$ is the decomposition into cyclic groups,

$$\mathcal{X}=\bigoplus_{i=1}^{r}\mathcal{X}_i \qquad \mathcal{X}_i=\times_{a\in\mathbb{L}}\mathcal{G}_i \qquad X=\sum_{i=1}^{r}X_i \qquad X_i\in\mathcal{X}_i.$$

Similarly for $\alpha_1\in\mathcal{M}$ and $\alpha_2\in\mathcal{M}_f$:

$$\langle\alpha_1,\alpha_2\rangle=\prod_{B\in\mathcal{B}}\exp[2\pi i\alpha_i(B)\alpha_2(B)/|B|]. \tag{2.13}$$

Proposition 2.1

(*a*) $\langle\gamma(X),\alpha\rangle=\langle X,\varepsilon(\alpha)\rangle$ where $X\in\mathcal{X}$, $\alpha\in\mathcal{M}_f$.
(*b*) $\langle\alpha,\gamma(X)\rangle=\langle\varepsilon(\alpha),X\rangle$ where $X\in\mathcal{X}_f$, $\alpha\in\mathcal{M}$.

Let $\mathcal{N}\subset\mathcal{M}_f(\mathcal{X}_f)$. Then

$$\mathrm{cl}\ \mathcal{N}\equiv\mathrm{Cl}\ \mathcal{N}\cap\mathcal{M}_f(\mathrm{Cl}\ \mathcal{N}\cap\mathcal{X}_f)$$

$$\Gamma_f^0\equiv\{f:f:\mathcal{M}_f\to\mathbb{Z}_m[\mathbb{Z}^\nu], f(\beta)=0 \text{ for every } \beta\in\Gamma_f\}.$$

3. Reduction

Let $(\mathbb{L},K)$, $(\mathbb{L}',K')$ be two lattice systems. Namely, $\mathbb{L}$ and $\mathbb{L}'$ are two lattices, $K(B)=\beta J(B)$, $K'(B')=\beta J'(B')$, where $J(B)$, $J'(B')$ are the coupling constants for bonds (characters), B and B' respectively, and β is the inverse temperature. Let ϕ be a continuous homomorphism of $\mathcal{X}$ to $\mathcal{X}'$ and let $\hat{\phi}:\mathcal{X}'_f\to\mathcal{X}_f$ be its dual. ϕ is a morphism from $(\mathbb{L},K)$ to $(\mathbb{L}',K')$ if it commutes with the action of $\mathbb{Z}^\nu$ and satisfies the following three conditions:

(i) $\phi(\mathcal{X}_f)\subset\mathcal{X}'_f$

(ii) $\hat{\phi}$ yields a bijection $\mathcal{B}'\to\mathcal{B}$ such that $|B'|=|\hat{\phi}(B')|$ and $K(\hat{\phi}(B'))=K'(B')$ for any $B'\in\mathcal{B}'$,

(iii) If $\phi: \mathcal{M} \to \mathcal{M}'$ is the isomorphism induced by $\hat{\phi}$ (i.e. if $\phi(\alpha) = \alpha'$, then $\alpha(\hat{\phi}(B')) = \alpha'(B')$ for any $B' \in \mathcal{B}'$), then $\phi(\Gamma_f)$ is dense in Γ'_f.

The following theorem relates the system to its reduced version. It is a direct generalisation of the corresponding theorem in [5].

Theorem 3.1. Let $(\mathbb{L}, K)$ and $(\mathbb{L}', K')$ be two ferromagnetic lattice systems and let ϕ be a morphism from $(\mathbb{L}, K)$ to $(\mathbb{L}', K')$. If the system $(\mathbb{L}', K')$ has the decomposition property, then

$$\phi\rho^+ = \rho'^+ \qquad \text{hence } \rho'^+(\hat{A}) = \rho^+(\widehat{\hat{\phi}(A)}) \text{ for all } A \in \mathcal{X}_f \tag{3.1}$$

$$\rho^+(\hat{A}) = 0 \text{ if } A \notin \hat{\phi}(\mathrm{cl}\, \mathcal{A}'). \tag{3.2}$$

The proof of the theorem follows the corresponding one in [5] and can also be found in [12].

In particular, theorem 2.1 says that $\mathcal{B}^+ = \hat{\phi}(\mathcal{B}'^+)$ and equivalently $\mathcal{S}^+ = \phi^{-1}(\mathcal{S}'^+)$. Since the system $(\mathbb{L}', K')$ has the decomposition property, $\mathcal{B}^+ = \hat{\phi}(\mathrm{cl}\, \mathcal{A}')$ at low temperatures. It can be proven [5] that $\hat{\phi}$ is an isomorphism between cl $\mathcal{A}'$ and $\mathcal{A}^{**}$ (the double dual of the module $\mathcal{A}$). The double dual of the inclusion map i, $i^{**}: \mathcal{A}^{**} \to \mathcal{X}_f$, is injective [13], hence $\mathcal{A}^{**}$ can be identified with its image in $\mathcal{X}_f$. Finally $\mathcal{B}^+ = \mathcal{A}^{**}$ at low temperatures so the number of pure phases is equal to $|\mathcal{A}^{**}/\mathcal{A}|$ at low temperatures.

4. Construction of the reduced system

Proposition 4.1. Let $\phi: \oplus_{i \in \mathbb{L}} \mathcal{G} \to \oplus_{i \in \mathbb{L}'} \mathcal{G}'$ be a $\mathbb{Z}_m[\mathbb{Z}^\nu]$-module homomorphism. Then ϕ extends to a unique continuous module homomorphism $\phi: \times_{i \in \mathbb{L}} \mathcal{G} \to \times_{i \in \mathbb{L}'} \mathcal{G}'$.

Proof. It is sufficient to consider the case of $\mathbb{L}, \mathbb{L}' = \mathbb{Z}^\nu$. The proposition follows from the following fact: for any finite subset $A \subset \mathbb{L}'$ there exists a finite subset $B \subset \mathbb{L}$ such that for any $x \in \oplus_{i \in \mathbb{Z}^\nu} \mathcal{G}$, $\underline{x} \cap B = \phi$ we have $\underline{\phi(x)} \cap A = \phi$ (continuity at $e_{\mathcal{X}}$), where $\underline{x} = \{i \in \mathbb{Z}^\nu;\, x(i) \neq e_{\mathcal{G}}\}$.

Let $y_k \in \oplus_{i \in \mathbb{Z}^\nu} \mathcal{G}$, $\underline{y}_k = \{0\}$. $y_k(0)$ is the generator of $\mathcal{G}$ and k runs over all generators of $\mathcal{G}$ in the decomposition of $\mathcal{G}$ into cyclic groups. For a finite $A \subset \mathbb{Z}^\nu = \mathbb{L}'$ let

$$B = \left\{ a \in \mathbb{Z}^\nu : \left(a \bigcup_k \underline{\phi(y_k)} \right) \cap A \neq \phi \right\}.$$

B has the required property.

Let $(\mathbb{L}, K)$ be a lattice system. The reduction of $(\mathbb{L}, K)$ can be constructed as follows.

Let F be a finite family of generators of $\mathrm{cl}\, \Gamma_f$ and let $\mathcal{G}' = \oplus_{\alpha \in F} \mathbb{Z}_{|\alpha|}$, where $|\alpha|$ is the order of α in $\mathcal{M}$. Such a family exists since $\mathrm{cl}\, \Gamma_f$ is a submodule of a finitely generated module $\mathcal{M}_f$ over a Noetherian ring $\mathbb{Z}_m[\mathbb{Z}^\nu]$.

Let $\mathcal{X}' = \times_{i \in \mathbb{Z}^\nu} \mathcal{G}'$ and let $\eta: \mathcal{X}'_f \to \mathrm{cl}\, \Gamma_f$ be the $\mathbb{Z}_m[\mathbb{Z}^\nu]$ homomorphism which extends the inclusion map $\{y'_k\} \to \mathrm{cl}\, \Gamma_f$, where y'_k are from proposition 4.1. By proposition 4.1 η has a unique extension to a continuous homomorphism $\eta: \mathcal{X}' \to \mathcal{M}$.

Proposition 4.2. $\eta(\mathcal{X}') = \gamma(\mathcal{X})$.

Proof. By construction $\eta(\mathcal{X}') = \mathrm{cl}\, \Gamma_f$. Let $\alpha = \gamma(X)$, $X = \Sigma_i X_i$, $X_i \in \mathcal{X}_f$ are such that $\underline{X}_i \cap \underline{X}_j = \phi$ if $i \neq j$, so $\gamma(X) = \Sigma_i \gamma(X_i)$. Let X'_i be such that $\eta(X'_i) = \gamma(X_i)$, then $\eta(\Sigma_i X'_i) = \gamma(X)$.

$\hat{\eta}$ denotes the dual map; $\hat{\eta} : \mathcal{M}_f \to \mathcal{X}_f$. Define $B' = \hat{\eta}(\alpha_B)$, $B \in \mathcal{B}$, $\alpha_B(A) = \delta_{A,B}$, $A \in \mathcal{B}$ and let $\mathcal{B}' = \{B' \colon B \in \mathcal{B}\}$.

The interaction K' on $\mathcal{B}'$ is defined by $K'(B') = K(B)$ for any $B \in \mathcal{B}$.

Proposition 4.3. The order of B' is equal to the order of B for any B from $\mathcal{B}_0$.

Proof. Obviously $|B'|$ divides $|B|$. For s, $0 < s < |B|$, there is $X \in \mathcal{X}$ such that $\langle X, \varepsilon(s\alpha_B)\rangle \neq 1$.

$$\langle X, \varepsilon(s\alpha_B)\rangle = \langle \gamma(X), s\alpha_B\rangle = \langle \eta(X'), s\alpha_B\rangle = \langle X', s\hat{\eta}(\alpha_B)\rangle = \langle X', sB'\rangle$$

where $X' \in \mathcal{X}'$ so $|B'| \neq s$ and finally $|B'| = |B|$.

Proposition 4.4. $B \to B'$ is a bijection of $\mathcal{B}$ onto $\mathcal{B}'$.

Proof. Let $\hat{\eta}(\alpha_{B_1}) = \hat{\eta}(\alpha_{B_2})$, $B_1, B_2 \in \mathcal{B}$, then for any $X' \in \mathcal{X}'$

$$\langle X', \hat{\eta}(\alpha_{B_1})\rangle = \langle X', \hat{\eta}(\alpha_{B_2})\rangle$$
$$\langle \eta(X'), \alpha_{B_1}\rangle = \langle \eta(X'), \alpha_{B_2}\rangle$$
$$\langle \gamma(X), \alpha_{B_1}\rangle = \langle \gamma(X), \alpha_{B_2}\rangle$$

for any $X \in \mathcal{X}$, so $\hat{B}_1(X) = \hat{B}_2(X)$ for any $X \in \mathcal{X}$, hence $B_1 = B_2$.

Proposition 4.5. Γ'_f is isomorphic to $\mathrm{cl}\,\Gamma_f$.

Proof. The proposition follows from the following equalities:

$$\langle X', \hat{\eta}(\alpha_B)\rangle = \langle \eta(X'), \alpha_B\rangle = \langle \gamma(X), \alpha_B\rangle = \langle X, \varepsilon(\alpha_B)\rangle = \langle X, B\rangle$$

where $X \in \mathcal{X}$, $X' \in \mathcal{X}'_f$.

By the criterion from [5, 8] the system $(\mathbb{L}', K')$ has the decomposition property. Now take $\{y^i_k\}_{i \in \mathbb{L}}$ from proposition 4.1 (y_k for each copy of $\mathbb{Z}^\nu$ in $\mathbb{L}$). Let $\{\gamma(y^i_k), \beta_j\}$ be the family of generators of $\mathrm{cl}\,\Gamma_f$. The homomorphism $\phi : \mathcal{X}_f \to \mathcal{X}'_f$ is constructed to make the following diagram commute:

$$\begin{array}{ccc} \mathcal{X}_f & \xrightarrow{\ \gamma\ } & \Gamma_f \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle \text{inclusion}} \\ \mathcal{X}'_f & \xrightarrow[\ \eta\ ]{} & \mathrm{cl}\,\Gamma_f \simeq \Gamma'_f \end{array} \qquad \eta\phi = \gamma.$$

Let $\phi : \mathcal{X} \to \mathcal{X}'$ be the extension which exists by proposition 4.1. ϕ defines a reduction of $(\mathbb{L}, K)$. The inclusion $\phi(\mathcal{X}_f) \subset \mathcal{X}'_f$ follows from the construction

$$\eta\phi = \gamma \qquad \hat{\gamma} = \varepsilon_{|\mathcal{U}_f} \qquad \text{so } \hat{\phi}\hat{\eta} = \varepsilon$$

and hence $\hat{\phi}$ yields the bijection $\mathcal{B}' \to \mathcal{B}$ as needed. Obviously $\phi(\Gamma_f)$ is dense in Γ_f.

5. An effective method of reduction

The process of determination of $\mathcal{B}^+$ and $\mathcal{S}^+$ in theorem 3.1 is non-unique. There exists generally no canonical choice of $\mathbb{L}'$ and ϕ for constructing $\hat{\phi}(\mathrm{cl}\,\mathcal{A}') = \mathcal{B}^+$. However, there is a natural reduction in the case of $\mathbb{L} = \mathbb{Z}^\nu$ and $\mathcal{G} = \mathbb{Z}_m$, $m = \prod_{l=1}^k p_l$, where p_l are different prime numbers. We begin with the following proposition which is true for arbitrary m.

Proposition 5.1. Γ_f is a free $\mathbb{Z}_m[\mathbb{Z}^\nu]$-module generated by one element.

Proof. Let $\gamma_0 = \gamma(X^0)$, where $X^0(a) = \delta_{0,a}$, $0, a \in \mathbb{Z}^\nu$. γ_0 generates Γ_f. If $R \in \mathbb{Z}_m[\mathbb{Z}^\nu]$, $R \neq 0$, then $R\gamma_0 = \gamma(R) \neq 0$ because γ is injective when restricted to $\mathscr{X}_f$ [4].

For m described above we have the following proposition.

Proposition 5.2. $\mathrm{cl}\,\Gamma_f$ is a free $\mathbb{Z}_m[\mathbb{Z}^\nu]$-module generated by one element.

Proof. Let $\mathscr{B}_0 = \{B_1, \ldots, B_n\}$ and for $1 \leq i < j \leq n$

$$\beta_{ij} = B_i\alpha_{B_j} - B_j\alpha_{B_i}$$

where $\alpha_B \in \mathscr{M}$, $\alpha_B(A) = \delta_{B,A}$, $A \in \mathscr{B}$. For $R \in \mathbb{Z}_m[\mathbb{Z}^\nu]$ let $I(R)(a) = -R(a)$. It is easy to see that

$$\gamma_0 = \sum_{B \in \mathscr{B}} (|B|/m)I(B)\alpha_B.$$

Now we introduce

$$\gamma_0' = \sum_{B \in \mathscr{B}} (|B|/m)I(B/D)\alpha_B$$

where D is the greatest common divisor (GCD) of $\mathscr{B}_0$ and D is not a zero divisor (cf [8]). To show that $\gamma_0' \in \mathrm{cl}\,\Gamma_f$ let $f \in (\Gamma_f)^0$, then

$$I(D)f(\gamma_0') = f(I(D)\gamma_0') = f(\gamma_0) = 0$$

hence $f(\gamma_0') = 0$. Let $\alpha \in \mathrm{cl}\,\Gamma_f$,

$$\alpha = \sum_{i=1}^{n} (|B_i|/m)P_i\alpha_{B_i} \qquad \text{where } P_i \in \mathbb{Z}_m[\mathbb{Z}^\nu]$$

$$P_i = \sum_{l=1}^{k} (m/p_l)P_i^l \qquad \text{where } P_i^l \in \mathbb{Z}_{p_l}[\mathbb{Z}^\nu].$$

$\beta_{ij} \in \mathscr{K}_f$, so $\langle \beta_{ij}, \alpha \rangle = 0$, hence $P_iI(B_j) = P_jI(B_i)$, $1 \leq i < j \leq n$ (cf [4, 8]).

$P_i^l(I(B_j^l)/I(D^l)) = P_j^l(I(B_i^l)/I(D^l))$ in $Z_{p_l}[\mathbb{Z}^\nu]$ for every l; $1 \leq l \leq k$, so $P_i^l = P_j^l \equiv P^l$, hence $P_i^l = P^l(I(B_i^l)/I(D^l))$.

$$\alpha = \sum_{i=1}^{n} (|B_i|/m) \sum_{l=1}^{k} (m/p_l)P^l(I(B_i^l)/I(D^l))\alpha_i$$

so

$$\alpha = P\sum_{i=1}^{n} (|B_i|/m)(I(B_i)/I(D))\alpha_i = P\gamma_0'$$

where

$$P = \sum_{l=1}^{k} (m/p_l)P^l \qquad \text{and similarly} \qquad D = \sum_{l=1}^{k} (m/p_l)D^l$$

$$B_i = \sum_{l=1}^{k} (m/p_l)B_i^l \qquad P^l, D^l, B_i^l \in \mathbb{Z}_{p_l}[\mathbb{Z}^\nu].$$

It follows that $\mathrm{cl}\,\Gamma_f$ is generated by γ_0'. To see that $\mathrm{cl}\,\Gamma_f$ is free let $R \in \mathbb{Z}_m[\mathbb{Z}^\nu]$. $R\gamma_0' = 0$ means that $RI(D)\gamma_0' = R\gamma_0 = 0$, so $R = 0$.

The main concern now is to find generators of $\mathrm{cl}\,\Gamma_f$ in the cases of non-unique reduction. We discuss first the case of $\mathbb{L}=\mathbb{Z}^\nu$, $\mathscr{G}=Z_{p^2}$. Let $\mathscr{A}^p=\{A\in\mathscr{A}: pA=0\}$. $\gamma^{\mathscr{A}^p}(X)$ is a contour for the $\mathscr{A}^p$ system for a fixed choice of generators of $\mathscr{A}^p$ as bonds; similarly $\gamma^{p\mathscr{A}}(X)$. By proposition 5.2 $\mathrm{cl}\,\Gamma_f^{\mathscr{A}^p}$ is generated by $\gamma^{\mathscr{A}^p}(X)$ and $\mathrm{cl}\,\Gamma_f^{p\mathscr{A}}$ is generated by $\gamma^{p\mathscr{A}}(Y)$, where $X, Y\in\times_{i\in\mathbb{Z}^\nu}\mathbb{Z}_p$ and both modules are $\mathbb{Z}_p[\mathbb{Z}^\nu]$-modules.

Lemma 5.3. If there is a generator $\gamma^{\mathscr{A}^p}(X)$, of $\mathrm{cl}\,\Gamma_f^{\mathscr{A}^p}$, such that $\gamma(X)$ is finite, then $\mathrm{cl}\,\Gamma_f$ is generated by $\gamma(X)$ and $\gamma(pY)$, where $\gamma(Y)$ is a generator of $\mathrm{cl}\,\Gamma_f^{pA}$. The coefficients can be taken from $\mathbb{Z}_p[\mathbb{Z}^\nu]$.

Proof. Let $\gamma(Z)$ be finite, so $\gamma^{\mathscr{A}^p}(Z)$ is finite, hence

$$\gamma^{\mathscr{A}^p}(Z)=R\gamma^{\mathscr{A}^p}(X)\qquad R\in\mathbb{Z}_p[\mathbb{Z}^\nu].$$

Now it can be proven that $Z=RX+S_1+pX_1$, $S_1\in\mathscr{S}(\mathscr{A})$, $X_1\in\mathscr{X}$ [8]. Because $\gamma(pX_1)$ is finite, $\gamma^{p\mathscr{A}}(X_1)$ is finite, so $\gamma^{p\mathscr{A}}(X_1)=P\gamma^{p\mathscr{A}}(Y)$, where $P\in\mathbb{Z}_p[\mathbb{Z}^\nu]$, hence $X_1=PY+S_2$, where $S_2\in\mathscr{S}(p\mathscr{A})$, and finally $Z=RX+PY+S_1+pS_2$, so $\gamma(Z)=R\gamma(X)+P\gamma(pY)$.

The following two propositions describe cases for which the assumption of lemma 5.3 is satisfied.

Proposition 5.4. If $\mathscr{A}^p$ is reduced, then $\mathrm{cl}\,\Gamma_f$ is generated by $\gamma(X)$ and $\gamma(pY)$ as in lemma 5.3.

Proof. If $\mathscr{A}^p$ is reduced, then $\mathrm{cl}\,\Gamma_f^{\mathscr{A}^p}$ is equal to $\Gamma_f^{\mathscr{A}^p}$ and is generated by $\gamma^{\mathscr{A}^p}(X^0)$, where $X^0(a)=\delta_{0,a}$, $0, a\in\mathbb{Z}^\nu$.

Proposition 5.5. If $p\mathscr{A}$ is the principal ideal in $\mathbb{Z}_p[\mathbb{Z}^\nu]$, then $\mathrm{cl}\,\Gamma_f$ is generated by $\gamma(X)$ and $\gamma(pY)$ as in lemma 5.3.

Proof. Let $p\mathscr{A}$ be generated by pA. By proposition 5.2 there is $Z\in\mathscr{X}$ such that $pA(Z)=\exp(2\pi i/p)$ and $pA_i(z)=1$ for any translate pA_i of pA. Let X_1 be such that $\gamma^{\mathscr{A}^p}(X_1)$ is a generator of $\mathrm{cl}\,\Gamma_f^{\mathscr{A}^p}$. There is finite Λ such that if $B\in\mathscr{B}$, $pB\neq 0$ and $\underline{B}\cap\Lambda=\phi$, then $\gamma(X_1)(B)=p$. Now using the fact that Γ is closed we can find $Z_1\in p\mathscr{S}$ such that $\gamma(X_1+Z_1)$ is finite. Let $X=X_1+Z_1$. Obviously $\gamma^{\mathscr{A}^p}(X)$ is a generator of $\mathrm{cl}\,\Gamma_f^{\mathscr{A}^p}$.

6. Examples

We compute the number of extremal periodic Gibbs states in several models. It will be convenient to use the following 'polynomial' notation for the bonds of the system. Let $\mathbb{L}=\mathbb{L}_1\cup\ldots\cup\mathbb{L}_l$ be the decomposition of $\mathbb{L}$ into a sum of $\mathbb{Z}^\nu$ lattices. Thus any element $A\in\mathscr{X}_f=\oplus_{i\in\mathbb{L}}\mathbb{Z}_m$ will be identified with a sequence of l elements of $\oplus_{i\in\mathbb{Z}^\nu}\mathbb{Z}_m$; $A=(A_1,\ldots,A_l)$.

On the other hand, elements of $\oplus_{i\in\mathbb{Z}^\nu}\mathbb{Z}_m$ are identified with polynomials in the following variables: $x_1,\ldots,x_\nu$, $x_1^{-1},\ldots,x_\nu^{-1}$ with the coefficients in $\mathbb{Z}_m$ and $x_ix_i^{-1}=1$. Let

$$B\in\bigoplus_{i\in\mathbb{Z}^\nu}\mathbb{Z}_m\qquad \underline{B}=\{b_1,\ldots,b_k\}\qquad B(b_i)\equiv B_i\qquad b_i=(b_i^1,\ldots,b_i^\nu)\in\mathbb{Z}^\nu$$

then we can write:

$$B = \sum_{i=1}^{k} B_i \prod_{j=1}^{\nu} x_j^{b_j^i}.$$

In all the examples below, $\mathbb{Z}_4$ is placed at each site of the square lattice. Several examples with spin-$\frac{1}{2}$ on general $\mathbb{Z}^\nu$-invariant lattices were discussed by Slawny in [5]. From now on x denotes x_1 and y denotes x_2.

Example 6.1. $\mathscr{B}_0 = \{B_1, B_2, B_1^{-1}, B_2^{-1}\}$, where $B_1 = 2+2x$, $B_2 = 1+x+y+xy$. $2\mathscr{A}$ is generated by $2+2x+2y+2xy$ so the system is not reduced. $\mathscr{A}^2$ is generated by $2+2x$. $\mathrm{cl}\,\Gamma_f^{\mathscr{A}^2}$ is generated by $\gamma^{\mathscr{A}^2}(Z)$, where

$$Z = \sum_{n=0}^{\infty} x^{2n} + 3 \sum_{n=1}^{\infty} x^{2n-1}.$$

$\mathrm{cl}\,\Gamma_f^{2\mathscr{A}}$ is generated by $\gamma^{2\mathscr{A}}(W)$, where

$$W = \sum_{n,m=0}^{\infty} x^n y^m.$$

By lemma 5.3 $\mathrm{cl}\,\Gamma_f$ is generated by $\gamma(Z)$ and $\gamma(2W)$.

The lattice of the reduced system has a fundamental family which consists of two elements. $\mathbb{Z}_4$ is placed on one $\mathbb{Z}^2$ sublattice, $\mathbb{Z}_2$ on the other.

New bonds can be obtained by the method described in § 3:

$$B_1' = (2, 0) \qquad B_2' = (1+y, 1).$$

The $(\mathbb{L}', K')$ system is reduced, hence $\mathscr{B}'^{+} = \mathrm{cl}\,\mathscr{A}'$. It is easy to see that $\mathrm{cl}\,\mathscr{A}' = \mathscr{A}'$:

$$\mathscr{B}^{+} = \hat{\phi}(\mathscr{B}'^{+}) = \hat{\phi}(\mathscr{A}') = \mathscr{A}$$

hence at low temperatures there is a unique periodic Gibbs state.

Example 6.2. $\mathscr{B}_0 = \{B_1, B_2, B_3, B_1^{-1}, B_2^{-1}, B_3^{-1}\}$, where $B_1 = 1+x$, $B_2 = 1+3x$, $B_3 = 2+2y$.

$2\mathscr{A}$ is generated by $2+2x$ so the system is not reduced. $\mathscr{A}^2$ is generated by 2 so $\mathscr{A}^2$ is reduced. $\mathrm{cl}\,\Gamma_f^{\mathscr{A}^2}$ is generated by $\gamma^{\mathscr{A}^2}(1)$ and $\mathrm{cl}\,\Gamma_f^{2\mathscr{A}}$ is generated by $\gamma(Z)$, where $Z = \sum_{n=0}^{\infty} x^n$. $\mathrm{cl}\,\Gamma_f$ is generated by $\gamma(1)$ and $\gamma(2Z)$. Again, in the reduced system we have $\mathbb{Z}_4$ on one $\mathbb{Z}^2$ sublattice and $\mathbb{Z}_2$ on the other:

$$B_1' = (1+x, 1) \qquad B_2' = (1+3x, 1) \qquad B_3' = (2+2y, 0).$$

$\mathscr{B}'^{+}$ is generated by $(2, 0)$ and $(1+x, 1)$:

$$\hat{\phi}((2,0)) = 2 \qquad \hat{\phi}((1+x, 1)) = 1+x$$

hence $\mathscr{B}^{+} = \mathscr{A}$, so there is a unique periodic Gibbs state at low temperatures. The reason is that $\mathscr{A}$ can be generated by B_1 and B_2, so the system is essentially one dimensional.

Example 6.3. $\mathscr{B}_0 = \{B_1, B_2, B_1^{-1}, B_2^{-1}\}$, where $B_1 = 1+x+y^2+xy^2$, $B_2 = 1+x^2+y+x^2y$.

$2\mathscr{A}$ is generated by $2+2x+2y^2+2xy^2$ and $2+2x^2+2y+2x^2y$.

$\mathrm{GCD}[(1/2)2\mathscr{A}] = 1+x+y+xy$ so the system is not reduced. $\mathscr{A}^2$ is generated by $2+2x^2y^2$, $2x^2+2y^2$, $2x+2y+2x^2y+2xy^2$, $2x+2y^2+2xy^2+2x^2y^2$, $2+2y+2x^2y+2y^2$.

$\text{GCD}[(1/2)\mathscr{A}^2]=1$ so $\mathscr{A}^2$ is reduced. $\text{cl}\,\Gamma_f^{2,\mathscr{A}}$ is generated by $\gamma^{2,\mathscr{A}}(w)$, where

$$w=\sum_{n,m=0}^{\infty}x^n y^m.$$

$\text{cl}\,\Gamma_f$ is generated by $\gamma(1)$ and $\gamma(2w)$. In the reduced system $\mathbb{Z}_4$ is on the $\mathbb{Z}^2$ sublattice and $\mathbb{Z}_2$ on the other:

$$B_1'=(1+x^{-1}+y^{-2}+x^{-1}y^{-2},1+y)$$
$$B_2'=(1+x^{-2}+y^{-1}+x^{-2}y^{-1},1+x).$$

$\mathscr{B}'^{+}$ is generated by B_1', B_2' and $(2,0)$. $\hat{\phi}((2,0))=2$.

After some algebra it can be shown that $|\mathscr{B}^{+}/\mathscr{A}|=249$, so there are 249 extremal periodic Gibbs states at low temperatures.

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